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Generalized torsion of elastic cylinders with microstructure

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Abstract. In this paper we use the method established by Day [1] to solve Truesdell's problem rephrased for the torsion of elastic cylinders with microstructure.

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1 Introduction

Let K denote the set of all displacement fields that correspond to the solutions of the torsion problem. Truesdell [7-9] proposed the following problem: for an isotropic linearly elastic cylinder subject to end tractions equipolent to a torque M, define a functional $\tau(\cdot)$ on K such that $M = D\tau(\mathbf{u})$, for each $\mathbf{u} \in K$, where D depends only on the cross section and elasticity field. In [1], Day established an elegant solution of Truesdell's problem and called $\tau(\mathbf{u})$ the generalized twist at $\mathbf{u}$. Truesdell's problem can be set for the torsion of elastic cylinders with microstructure. The theory of media with microstructure was developed in various works (see [2-4,6]). The torsion problem for elastic cylinders with microstructure has been investigated in [5]. In this paper we use the method established by Day [1] to solve Truesdell's problem for inhomogeneous and anisotropic bodies with microstructure.

2 Basic Equations

Throughout this paper B denotes a bounded regular region of three-dimensional Euclidean space. We call ∂B the boundary of B, and designate by $\mathbf{n}$ the outward unit normal of ∂B. Throughout this paper a rectangular Cartesian coordinate system $Ox_k (k = 1, 2, 3)$ is used. Letters in boldface stand for tensors of an order $p \geq 1$, and if $\mathbf{v}$ has the order p, we write $v_{ij\ldots kl}$ (p subscripts) for the components of $\mathbf{v}$ in the rectangular Cartesian coordinate frame. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers $(1,2)$, where Latin subscripts-unless otherwise specified-are confined to the range $(1,2,3)$; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

Assume that B is occupied by a linearly elastic material with microstructure. Let u_i denote the components of the displacement vector field, and let φ_{ij} denote the components of the microdeformation tensor. We introduce the twelve-dimensional vector $u = (u_1, u_2, u_3, \varphi_{11}, \varphi_{22}, \dots, \varphi_{13}) = (u_i, \varphi_{jk})$. The strain measures associated with u are defined by

$$e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \gamma_{ij}(u) = u_{j,i} - \varphi_{ij}, \quad \kappa_{ijk}(u) = \varphi_{jk,i}, \tag{1}$$

where e_{ij} is the macrostrain tensor, γ_{ij} is the relative deformation tensor and κ_{ijk} is the microdeformation gradient tensor [3,6]. The constitutive equations appropriate to the linearized theory of elasticity are

$$\begin{aligned} \tau_{ij}(u) &= C_{ijrs}e_{rs}(u) + G_{rsij}\gamma_{rs}(u) + F_{pqrij}\kappa_{pqr}(u), \\ \sigma_{ij}(u) &= G_{ijrs}e_{rs}(u) + B_{rsij}\gamma_{rs}(u) + D_{ijpqr}\kappa_{pqr}(u), \\ \mu_{ijk}(u) &= F_{ijkrs}e_{rs}(u) + D_{rsijk}\gamma_{rs}(u) + A_{ijkpqr}\kappa_{pqr}(u), \end{aligned} \tag{2}$$

where $\tau_{ij}(u)$ denotes the stress tensor, $\sigma_{ij}(u)$ means the relative stress tensor, $\mu_{ijk}(u)$ is the double stress tensor associated with u, and $A_{ijkpqr}, B_{ijrs}, \dots,$ G_{ijrs} are constitutive coefficients.

We call a vector field $u = (u_i, \varphi_{jk})$ an equilibrium vector field for B if $u_i, \varphi_{jk} \in C^1(\overline{B}) \cap C^2(B)$ and

$$[\tau_{ij}(u) + \sigma_{ij}(u)]_{,i} = 0, \quad (\mu_{ijk}(u))_{,i} + \sigma_{jk}(u) = 0, \tag{3}$$

hold on B. The traction and the double-traction at regular points of ∂B corresponding to u are defined by

$$T_i(u) = (\tau_{ji}(u) + \sigma_{ji}(u))n_j, \quad M_{ij}(u) = \mu_{rij}(u)n_r. \tag{4}$$

The strain energy density per unit volume corresponding to u is given by

$$\begin{aligned} \varepsilon(u) = {} & \frac{1}{2}C_{ijrs}e_{ij}(u)e_{rs}(u) + \frac{1}{2}B_{ijrs}\gamma_{ij}(u)\gamma_{rs}(u) + \\ & + \frac{1}{2}A_{ijkrmn}\kappa_{ijk}(u)\kappa_{rmn}(u) + D_{ijkrm}\gamma_{ij}(u)\kappa_{krm}(u) + \\ & + F_{ijkrm}\kappa_{ijk}(u)e_{rm}(u) + G_{ijkr}\gamma_{ij}(u)e_{kr}(u), \end{aligned} \tag{5}$$

where $A_{ijkrmn}, B_{ijrs}, C_{ijrs}, D_{ijkrm}, F_{ijkrm}$ and G_{ijkr} are smooth functions on $\overline{B}$ such that

$$\begin{aligned} & A_{ijkrmn} = A_{rmnijk}, \quad B_{ijrs} = B_{rsij}, \quad C_{ijrs} = C_{rsij}, \\ & F_{ijkrs} = F_{ijksr}, \quad G_{ijrs} = G_{ijsr}. \end{aligned} \tag{6}$$

We assume that the strain energy density is a positive definite quadratic form in the components of the strain measures.

The strain energy $E(u)$ corresponding to a smooth vector field u on B is

$$E(u) = \int_B \varepsilon(u) dv. \tag{7}$$

The functional $E(\cdot)$ generates the bilinear functional

$$\begin{aligned} E(u,v) = \frac{1}{2}\int_B \{ & C_{ijrs} e_{ij}(u) e_{rs}(v) + B_{ijrs}\gamma_{ij}(u)\gamma_{rs}(v) + \\ & + A_{ijkrmn}\kappa_{ijk}(u)\kappa_{rmn}(v) + D_{ijkrs}[\gamma_{ij}(u)\kappa_{krm}(v) + \\ & + \gamma_{ij}(v)\kappa_{krm}(u)] + F_{ijkrm}[\kappa_{ijk}(u)e_{rm}(v) + \\ & + \kappa_{ijk}(v)e_{rm}(u)] + G_{ijkr}[\gamma_{ij}(u)e_{kr}(v) + \gamma_{ij}(v)e_{kr}(u)]\} dv. \end{aligned} \tag{8}$$

We introduce the notations

$$< u, v > = 2E(u,v), \quad \|u\|_e^2 = < u, u > . \tag{9}$$

For any equilibrium vector fields $u = (u_i, \varphi_{jk})$ and $v = (v_i, \psi_{jk})$ one has

$$< u, v > = \int_{\partial B} [v_i T_i(u) + \psi_{jk} M_{jk}(u)] da, \tag{10}$$

and

$$\int_{\partial B} [u_i T_i(v) + \varphi_{jk} M_{jk}(v)] da = \int_{\partial B} [v_i T_i(u) + \psi_{jk} M_{jk}(u)] da. \tag{11}$$

Following [1], for any given equilibrium vector fields $u, v^{(1)}, v^{(2)}, v^{(3)}$ and $v^{(4)}$ we define the real function f of the variables ξ_1, ξ_2, ξ_3 and ξ_4 by

$$f = \|u - \sum_{s=1}^{4} \xi_s v^{(s)}\|_e^2. \tag{12}$$

In the following section the vector field u will be a solution of a certain boundary-value problem and the equilibrium vector fields $v^{(s)}, (s = 1, 2, 3, 4)$, will be prescribed. We have

$$f = \sum_{r,s=1}^{4} A_{rs}\xi_r\xi_s - 2\sum_{s}^{4} \xi_s < u, v^{(s)} > + \|u\|_e^2, \tag{13}$$

where

$$A_{rs} = < v^{(r)}, v^{(s)} >, \quad (r, s = 1, 2, 3, 4). \tag{14}$$

Since the matrix (A_{rs}) is positive definite, f will be a minimum at $(\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u))$ if and only if $\alpha_1(u), \alpha_2(u), \alpha_3(u)$ and $\alpha_4(u)$ satisfy the equations

$$\sum_{s=1}^{4} A_{rs}\alpha_s(u) =< u, v^{(r)} >, \quad (r = 1, 2, 3, 4). \tag{15}$$

In order to extend the result of [1] to the case of bodies with microstructure, we rephrase Truesdell's problem in the following manner: for a linearly elastic cylinder subject to end tractions equipolent to a torque M, define the quantities $\tau_s, (s = 1, 2, 3, 4)$, in such a way that

$$M\delta_{r4} = \sum_{s=1}^{4} D_{rs}\tau_s, \quad (r = 1, 2, 3, 4), \tag{16}$$

where δ_{pq} is the Kronecker delta, and $D_{rs}, (r, s = 1, 2, 3, 4)$, depend only the cross section and the constitutive coefficients.

3 Generalized Torsion

Assume that the region B from here on refers to the interior of a right cylinder of length h with the open cross section Σ and the lateral boundary Π. We denote by L the boundary of the generic cross section Σ. The rectangular Cartesian coordinate is chosen such that the x_3 axis is parallel to the generators of B and the x_1Ox_2 plane contains one of the terminal cross sections. We denote by Σ_1 and Σ_2, respectively, the cross section located at $x_3 = 0$ and $x_3 = h$. In view of the foregoing agreements we have

$$B = \{\mathbf{x}|(x_1, x_2) \in \Sigma, 0 < x_3 < h\}, \quad \Pi = \{\mathbf{x}|(x_1, x_2) \in L, 0 \le x_3 \le h\},$$
$$\Sigma_1 = \{\mathbf{x}|(x_1, x_2) \in \Sigma, x_3 = 0\}, \qquad \Sigma_2 = \{\mathbf{x}|(x_1, x_2) \in \Sigma, x_3 = h\},$$

where $\mathbf{x} = (x_1, x_2, x_3)$.

We assume for the remainder of this paper that the functions A_{ijkrmn}, $B_{ijrs}, C_{ijrs}, D_{ijkrs}, F_{ijkrm}, G_{ijrs}$ are independent of the axial coordinate and belong to $C^\infty(\overline{\Sigma}_1)$. Moreover, we assume that Σ_1 is C^∞-smooth.

We denote by $\mathbf{R}(u)$ and $\mathbf{H}(u)$, respectively, the resultant force and the resultant moment about O of the tractions associated with u, acting on Σ_2, i.e.,

$$\begin{aligned}
R_i(u) &= \int_{\Sigma_2} [\tau_{3i}(u) + \sigma_{3i}(u)]da, \\
H_\alpha(u) &= \int_{\Sigma_2} \varepsilon_{\alpha\beta}\{x_\beta[\tau_{33}(u) + \sigma_{33}(u)] + \mu_{3\beta3}(u) - \mu_{33\beta}(u)\}da, \\
H_3(u) &= \int_{\Sigma_2} \varepsilon_{\alpha\beta}\{x_\alpha[\tau_{3\beta}(u) + \sigma_{3\beta}(u)] + \mu_{3\alpha\beta}(u)\}da,
\end{aligned} \tag{17}$$

where $\varepsilon_{\alpha\beta}$ is the two-dimensional alternating symbol.

By a solution of the generalized torsion problem we mean an equilibrium vector field u that satisfies the conditions

$$[\tau_{\alpha i}(u)+\sigma_{\alpha i}(u)]n_\alpha = 0, \quad \mu_{\alpha ij}(u)n_\alpha = 0 \text{ on } \Pi, \tag{18}$$

$$R_i(u)=0, \quad H_\alpha(u)=0, \quad H_3(u)=M, \tag{19}$$

$$\begin{gathered}[\tau_{3j}(u)+\sigma_{3j}(u)](x_1,x_2,0)=[\tau_{3j}(u)+\sigma_{3j}(u)](x_1,x_2,h),\\ [\mu_{3jk}(u)](x_1,x_2,0)=[\mu_{3jk}(u)](x_1,x_2,h),\end{gathered} \tag{20}$$

where M is a prescribed constant.

Let Q denote the set of all equilibrium vector fields u that satisfy the conditions (18)-(20).

In what follows we will have occasion to use some results concerning the generalized plane strain problem for bodies with microstructure [5].

The state of generalized plane strain of B is characterized by

$$u_i=u_i(x_1,x_2), \quad \varphi_{jk}=\varphi_{jk}(x_1,x_2), \quad (x_1,x_2)\in\Sigma_1. \tag{21}$$

It follows from (1) and (21) that $e_{33}(u)=0, \kappa_{3jk}(u)=0$ and

$$\begin{gathered} e_{\alpha\beta}(u)=\frac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha}), \quad e_{\alpha 3}(u)=\frac{1}{2}u_{3,\alpha},\\ \gamma_{\alpha i}(u)=u_{i,\alpha}-\varphi_{\alpha i}, \quad \gamma_{3i}(u)=-\varphi_{3i}, \quad \kappa_{\alpha jk}(u)=\varphi_{jk,\alpha}.\end{gathered} \tag{22}$$

By (2) and (22),

$$\begin{aligned} \tau_{\alpha i}(u)&=C_{\alpha ij\beta}e_{j\beta}(u)+G_{kj\alpha i}\gamma_{kj}(u)+F_{\beta rs\alpha i}\kappa_{\beta rs}(u),\\ \sigma_{ij}(u)&=G_{ijr\beta}e_{r\beta}(u)+B_{krij}\gamma_{kr}(u)+D_{ij\beta rs}\kappa_{\beta rs}(u),\\ \mu_{\alpha ij}(u)&=F_{\alpha ijr\beta}e_{r\beta}(u)+D_{rs\alpha ij}\gamma_{rs}(u)+A_{\alpha ij\beta rs}\kappa_{\beta rs}(u).\end{aligned} \tag{23}$$

The equations of equilibrium (3), in the presence of the body force f_i and body double-force L_{ij}, take the form

$$(\tau_{\alpha j}(u)+\sigma_{\alpha j}(u))_{,\alpha}+f_i=0, \;\; (\mu_{\alpha ij}(u))_{,\alpha}+\sigma_{ij}(u)+L_{ij}=0. \tag{24}$$

We assume that on the lateral boundary we have the conditions

$$(\tau_{\alpha i}(u)+\sigma_{\alpha i}(u))n_\alpha=P_i, \;\; \mu_{\alpha ij}(u)n_\alpha=Q_{ij}, \tag{25}$$

where P_i and Q_{ij} are prescribed functions.

Clearly, the state of generalized plane strain demands that f_i, L_{ij}, P_i and Q_{ij} be independent of the axial coordinate.

The generalized plane strain problem consists in finding a vector field $u \in C^1(\overline{\Sigma}_1) \cap C^2(\Sigma_1)$ which satisfies the equations (24) on Σ_1 and the boundary conditions (25) on Γ.

The functions $\tau_{3i}(u)$ and $\mu_{3ij}(u)$ can be calculated after the determination of u.

The conditions of equilibrium for the cylinder B are

$$\begin{aligned} &\int_{\Sigma_1} f_i da + \int_\Gamma P_i ds = 0, \\ &\int_{\Sigma_1} \varepsilon_{\alpha\beta}(x_\alpha f_\beta + L_{\alpha\beta}) da + \int_\Gamma \varepsilon_{\alpha\beta}(x_\alpha P_\beta + Q_{\alpha\beta}) ds = 0 \end{aligned} \tag{26}$$

and

$$\begin{aligned} &\int_{\Sigma_1} (x_2 f_3 + L_{23} - L_{32}) da + \int_\Gamma (x_2 P_3 + Q_{23} - Q_{32}) ds - \\ &\qquad - \int_{\Sigma_1} (\tau_{32}(u) + \sigma_{32}(u)) da = 0, \\ &\int_{\Sigma_1} (x_1 f_3 + L_{13} - L_{31}) da + \int_\Gamma (x_1 P_3 + Q_{13} - Q_{31}) ds - \\ &\qquad - \int_{\Sigma_1} (\tau_{31}(u) + \sigma_{31}(u)) da = 0. \end{aligned} \tag{27}$$

The conditions (27) are identically satisfied on the basis of (24) and (25). Indeed, we have

$$\begin{aligned} &\int_{\Sigma_1} (\tau_{32}(u) + \sigma_{32}(u)) da = \int_{\Sigma_1} [\tau_{23}(u) + \sigma_{23}(u) + \sigma_{32}(u) - \sigma_{23}(u)] da = \\ &\quad = \int_{\Sigma_1} [\tau_{23}(u) + \sigma_{23}(u) + x_2\{\tau_{\alpha 3}(u)_{,\alpha} + \sigma_{\alpha 3}(u)_{,\alpha} + f_3\} + \\ &\qquad + L_{23} - L_{32} + (\mu_{\alpha 23}(u) - \mu_{\alpha 32}(u))_{,\alpha}] da = \\ &\quad = \int_\Sigma \{[x_2(\tau_{\alpha 3}(u) + \sigma_{\alpha 3}(u))]_{,\alpha} + x_2 f_3 + L_{23} - \\ &\qquad - L_{32} + (\mu_{\alpha 23}(u) - \mu_{\alpha 32}(u))_{,\alpha}\} da = \\ &\quad = \int_\Gamma (x_2 P_3 + Q_{23} - Q_{32}) ds + \int_{\Sigma_1} (x_2 f_3 + L_{23} - L_{32}) da. \end{aligned}$$

In a similar way we can prove that the second condition from (27) is satisfied.

It is known that [5] the boundary-value problem (24), (25) has a solution belonging to $C^\infty(\overline{\Sigma}_1)$ if and only if the C^∞ functions f_i, L_{ij}, P_i and Q_{ij} satisfy the conditions (26).

In what follows we will use four special problems $A^{(s)}, (s = 1, 2, 3, 4)$, of generalized plane strain for the domain Σ_1. The problem $A^{(s)}$ corresponds to the system of loading $\{f_i^{(s)}, L_{ij}^{(s)}, P_i^{(s)}, Q_{ij}^{(s)}\}$ where

$$
\begin{aligned}
f_i^{(\beta)} &= [(C_{\alpha i33} + G_{33\alpha i} + G_{\alpha i33} + B_{33\alpha i})\varepsilon_{\beta\nu}x_\nu + (D_{\alpha i3mn} + F_{3mn\alpha i})\varepsilon_{mn\beta}]_{,\alpha},\\
f_i^{(3)} &= [(C_{\alpha i\rho 3} + G_{3\rho\alpha i} + G_{\alpha i\rho 3} + B_{3\rho\alpha i})\varepsilon_{\beta\rho}x_\beta + (D_{\alpha i3\rho\nu} + F_{3\rho\nu\alpha i})\varepsilon_{\rho\nu}]_{,\alpha},\\
f_i^{(4)} &= (C_{\alpha i33} + G_{33\alpha i} + G_{\alpha i33} + B_{33\alpha i})_{,\alpha},\\
L_{ij}^{(\beta)} &= [(F_{\alpha ij33} + D_{33\alpha ij})\varepsilon_{\beta\nu}x_\nu + A_{\alpha ij3mn}\varepsilon_{mn\beta}]_{,\alpha} + (G_{ij33} + B_{33ij})\varepsilon_{\beta\nu}x_\nu +\\
&\quad + D_{ij3mn}\varepsilon_{mn\beta},\\
L_{ij}^{(3)} &= [(F_{\alpha ij\rho 3} + D_{3\rho\alpha ij})\varepsilon_{\beta\rho}x_\beta + A_{\alpha ij3\eta\rho}\varepsilon_{\eta\rho}]_{,\alpha} + (B_{3\alpha ij} + G_{ij\alpha 3})\varepsilon_{\beta\alpha}x_\beta +\\
&\quad + D_{ij3\alpha\beta}\varepsilon_{\beta\alpha},\\
L_{ij}^{(4)} &= (F_{\alpha ij33} + D_{33\alpha ij})_{,\alpha} + G_{ij33} + B_{33ij},\\
P_i^{(\beta)} &= [(C_{\alpha i33} + G_{33\alpha i} + G_{\alpha i33} + B_{33\alpha i})\varepsilon_{\nu\beta}x_\nu + (D_{\alpha i3mn} + F_{3mn\alpha i})\varepsilon_{nm\beta}]n_\alpha,\\
P_i^{(3)} &= [(C_{\alpha i\rho 3} + G_{3\rho\alpha i} + G_{\alpha i\rho 3} + B_{3\rho\alpha i})\varepsilon_{\rho\beta}x_\beta + (D_{\alpha i3\rho\nu} + F_{3\rho\nu\alpha i})\varepsilon_{\nu\rho}]n_\alpha,\\
P_i^{(4)} &= -(C_{\alpha i33} + G_{33\alpha i} + G_{\alpha i33} + B_{33\alpha i})n_\alpha,\\
Q_{ij}^{(\beta)} &= [(F_{\alpha ij33} + D_{33\alpha ij})\varepsilon_{\nu\beta}x_\nu + A_{\alpha ij3mn}\varepsilon_{nm\beta}]n_\alpha,\\
Q_{ij}^{(3)} &= [(F_{\alpha ij\rho 3} + D_{3\rho\alpha ij})\varepsilon_{\rho\beta}x_\beta + A_{\alpha ij3\rho\nu}\varepsilon_{\nu\rho}]n_\alpha,\\
Q_{ij}^{(4)} &= -(F_{\alpha ij33} + D_{33\alpha ij})n_\alpha,
\end{aligned}
\tag{28}
$$

where ε_{ijk} is the alternating symbol. It is a simple matter to verify that the necessary and sufficient conditions (26) for the existence of a solution are satisfied for each boundary-value problem $A^{(s)}$.

We denote by $w^{(s)} = (w_i^{(s)}, \omega_{ij}^{(s)})$ the solution of the problem $A^{(s)}$. Thus, the vector fields $w^{(s)}, (s = 1, 2, 3, 4)$, are characterized by

$$
\begin{aligned}
&[\tau_{\alpha j}(w^{(s)}) + \sigma_{\alpha j}(w^{(s)})]_{,\alpha} + f_i^{(s)} = 0,\\
&(\mu_{\alpha ij}(w^{(s)}))_{,\alpha} + \sigma_{ij}(w^{(s)}) + L_{ij}^{(s)} = 0 \ \text{ on } \Sigma_1,\\
&[\tau_{\alpha i}(w^{(s)}) + \sigma_{\alpha i}(w^{(s)})]n_\alpha = P_i^{(s)},\ \mu_{\alpha ij}(w^{(s)})n_\alpha = Q_{ij}^{(s)} \text{ on } \Gamma.
\end{aligned}
\tag{29}
$$

In what follows we assume that the vector fields $w^{(s)}, (s = 1, 2, 3, 4)$ are known. We note that $w^{(s)}$ depend only on the domain Σ_1 and the constitutive coefficients.

We define the vector fields $v^{(s)} = (v_i^{(s)}, \psi_{jk}^{(s)})$ on B, $(s = 1, 2, 3, 4)$, by

$$\begin{gathered} v_\alpha^{(\beta)} = \frac{1}{2}\varepsilon_{\alpha\beta}x_3^2 + w_\alpha^{(\beta)}, \quad v_3^{(\beta)} = \varepsilon_{\beta\alpha}x_\alpha x_3 + w_3^{(\beta)}, \\ v_\alpha^{(3)} = -\varepsilon_{\alpha\beta}x_\beta x_3 + w_\alpha^{(3)}, \quad v_3^{(3)} = w_3^{(3)}, \\ v_\alpha^{(4)} = w_\alpha^{(4)}, \quad v_3^{(4)} = x_3 + w_3^{(4)}, \\ \psi_{jk}^{(s)} = \varepsilon_{jks}x_3 + \omega_{jk}^{(s)}, \quad \psi_{jk}^{(4)} = \omega_{jk}^{(4)}. \end{gathered} \tag{30}$$

It follows from (2) and (30) that

$$\begin{aligned} \tau_{ij}(v^{(\beta)}) &= (C_{ij33} + G_{33ij})\varepsilon_{\beta\nu}x_\nu + F_{3mnij}\varepsilon_{mn\beta} + \tau_{ij}(w^{(\beta)}), \\ \tau_{ij}(v^{(3)}) &= (C_{ij\alpha 3} + G_{3\alpha ij})\varepsilon_{\beta\alpha}x_\beta + F_{3\rho\nu ij}\varepsilon_{\rho\nu} + \tau_{ij}(w^{(3)}), \\ \tau_{ij}(v^{(4)}) &= C_{ij33} + G_{33ij} + \tau_{ij}(w^{(4)}), \\ \sigma_{ij}(v^{(\beta)}) &= (G_{ij33} + B_{33ij})\varepsilon_{\beta\nu}x_\nu + D_{ij3mn}\varepsilon_{mn\beta} + \sigma_{ij}(w^{(\beta)}), \\ \sigma_{ij}(v^{(3)}) &= (B_{3\alpha ij} + G_{ij\alpha 3})\varepsilon_{\beta\alpha}x_\beta + D_{ij3\rho\nu}\varepsilon_{\rho\nu} + \sigma_{ij}(w^{(3)}), \\ \sigma_{ij}(v^{(4)}) &= G_{ij33} + B_{33ij} + \sigma_{ij}(w^{(4)}), \\ \mu_{ijk}(v^{(\beta)}) &= (F_{ijk33} + D_{33ijk})\varepsilon_{\beta\nu}x_\nu + A_{ijk3mn}\varepsilon_{mn\beta} + \mu_{ijk}(w^{(\beta)}), \\ \mu_{ijk}(v^{(3)}) &= (F_{ijk\alpha 3} + D_{3\alpha ijk})\varepsilon_{\beta\alpha}x_\beta + A_{ijk3\rho\nu}\varepsilon_{\rho\nu} + \mu_{ijk}(w^{(3)}), \\ \mu_{ijk}(v^{(4)}) &= F_{ijk33} + D_{33ijk} + \mu_{ijk}(w^{(4)}). \end{aligned} \tag{31}$$

On the basis of (29) we find that the vector fields $v^{(s)}, (s = 1, 2, 3, 4)$ satisfy the equations

$$[\tau_{ij}(v^{(s)}) + \sigma_{ij}(v^{(s)})]_{,i} = 0, \quad [\mu_{ijk}(v^{(s)})]_{,i} + \sigma_{jk}(v^{(s)}) = 0, \tag{32}$$

on B, and the conditions

$$[\tau_{\alpha i}(v^{(s)}) + \sigma_{\alpha i}(v^{(s)})]n_\alpha = 0, \quad \mu_{\alpha ij}(v^{(s)})n_\alpha = 0 \ \text{ on } \Gamma. \tag{33}$$

In view of (32) and (33), we get

$$\int_{\Sigma_2} [\tau_{3\alpha}(v^{(s)}) + \sigma_{3\alpha}(v^{(s)})]dv = 0. \tag{34}$$

The first of (34) follows from the relations

$$\begin{aligned} &\int_{\Sigma_2} (\tau_{31} + \sigma_{31})da = \int_{\Sigma_2} (\tau_{13} + \sigma_{13} + \sigma_{31} - \sigma_{13})da = \\ &\quad = \int_{\Sigma_2} [\tau_{13} + \sigma_{13} + x_1(\tau_{\alpha 3,\alpha} + \sigma_{\alpha 3,\alpha} + \tau_{33,3} + \sigma_{33,3}) + \mu_{i13,i} - \mu_{i31,i}]da = \\ &\quad = \int_{\Gamma} [x_1(\tau_{\alpha 3} + \sigma_{\alpha 3})n_\alpha + (\mu_{\alpha 13} - \mu_{\alpha 31})n_\alpha]ds = 0. \end{aligned}$$

In a similar way we can prove the second relation of (34). Let $u = (u_i, \varphi_{jk}) \in Q$. By (9), (10), (18)-(20) and (30), we get

$$\begin{aligned}
< u, v^{(\alpha)} > &= \int_{\partial B} [v_i^{(\alpha)} T_i(u) + \psi_{jk}^{(\alpha)} M_{jk}(u)] da = \\
&= \int_{\Sigma_2} \{v_i^{(\alpha)} [\tau_{3i}(u) + \sigma_{3i}(u)] + \psi_{jk}^{(\alpha)} \mu_{3jk}(u)\} da - \\
&\quad - \int_{\Sigma_1} \{v_i^{(\alpha)} [\tau_{3i}(u) + \sigma_{3i}(u)] + \psi_{jk}^{(\alpha)} \mu_{3jk}(u)\} da = h\varepsilon_{\alpha\beta} H_\beta(u) = 0, \\
< u, v^{(3)} > &= hH_3(u), \quad < u, v^{(4)} >= hR_3(u) = 0.
\end{aligned} \tag{35}$$

On the other hand, by (11), (4), (31) and (33) we find that

$$< u, v^{(3)} >= \int_{\partial B} [u_i T_i(v^{(3)}) + \varphi_{jk} M_{jk}(v^{(3)})] da = E(u), \tag{36}$$

where

$$\begin{aligned}
E(u) = &\int_{\Sigma_2} \{u_i [\tau_{3i}(v^{(3)}) + \sigma_{3i}(v^{(3)})] + \varphi_{jk} \mu_{3jk}(v^{(3)})\} da - \\
&- \int_{\Sigma_1} \{u_i [\tau_{3i}(v^{(3)}) + \sigma_{3i}(v^{(3)})] + \varphi_{jk} \mu_{3jk}(v^{(3)})\} da.
\end{aligned} \tag{37}$$

We introduce the notations

$$\begin{aligned}
L_{\alpha s} &= \int_{\Sigma_1} \{x_\alpha [\tau_{33}(v^{(s)}) + \sigma_{33}(v^{(s)})] + \mu_{3\alpha 3}(v^{(s)}) - \mu_{33\alpha}(v^{(s)})\} da, \\
L_{3s} &= \int_{\Sigma_1} [\tau_{33}(v^{(s)}) + \sigma_{33}(v^{(s)})] da, \\
L_{4s} &= \int_{\Sigma_1} \varepsilon_{\alpha\beta} \{x_\alpha [\tau_{3\beta}(v^{(s)}) + \sigma_{3\beta}(v^{(s)})] + \mu_{3\alpha\beta}(v^{(s)})\} da.
\end{aligned} \tag{38}$$

It follows from (10), (11) and (30) that

$$\begin{gathered}
< v^{(\alpha)}, v^{(s)} >= h\varepsilon_{\alpha\beta} L_{\beta s}, \quad < v^{(3)}, v^{(s)} >= hL_{4s}, \\
< v^{(4)}, v^{(s)} >= hL_{3s}.
\end{gathered} \tag{39}$$

In this case, by (35)-(37) and (39), the system (15) becomes

$$\sum_{s=1}^{4} L_{rs} \tau_s = E(u) \delta_{r4}, \quad (r = 1, 2, 3, 4). \tag{40}$$

We note that $L_{rs}, (r, s = 1, 2, 3, 4)$, depend only on the cross section and the constitutive coefficients. The system (40) defines $\tau_j(\cdot), (j = 1, 2, 3, 4)$, on the set of all equilibrium vector fields that satisfy the conditions (18)-(20).

References

[1] W. A. DAY: *Generalized torsion: The solution of a problem of Truesdell's,* Arch. Rational Mech. Anal., **76**(1981), 283–288.

[2] A. C. ERINGEN, E. S. SUHUBI: *Nonlinear theory of simple microelastic solids*, Int. J. Engng. Sci., **2**(1964), I: 189–203; II: 389–404.

[3] A. C. ERINGEN: Microcontinuum Field Theories I. Foundations and Solids, Springer-Verlag, New York, Berlin, Heidelberg, 1999.

[4] A. E. GREEN, R. S. RIVLIN: *Multipolar continuum mechanics*, Arch. Rational Mech. Anal., **17**(1964), 113–147.

[5] D. IEŞAN: *Saint-Venant's problem for inhomogeneous and anisotropic elastic solids with microstructure*, Arch. Mechanics, **29**(1977), 419–430.

[6] R. D. MINDLIN: *Microstructure in linear elasticity*, Arch. Rational Mech. Anal., **16**(1964), 51–77.

[7] C. TRUESDELL: *The rational mechanics of materials-past present, future*, Appl. Mech. Reviews, **12**(1959), 75–80.

[8] C. TRUESDELL: *The rational mechanics of materials-past, present, future (Corrected and modified reprint of [7]),* pp. 225–236 of Applied Mechanics Surveys, Spartan Books, 1966.

[9] C. TRUESDELL: *Some challenges offered to analysis by rational thermomechanics*, pp. 495–603 of Contemporary Developments in Continuum Mechanics and Partial Differential Equations, G.M. de la Penha & L.A. Medeiros, Eds., North-Holland, 1978.